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Model of a Nonhomogeneous Medium Conducting Light

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We consider a ray of light propagating within a system of infinitely many adjoining rectangles in a plane with passages between any pair of neighboring ones. The ray is assumed to be reflected by the sides of the rectangles, and is refracted while passing from one rectangle to its neighbor. We prove that if the sizes of the rectangles or the coefficients of refraction inside them are random, then with probability 1 the ray reaches arbitrarily remote rectangles.

KEY WORDS: Geometric optics; random media; billiards.

INTRODUCTION

In order to analyze accurately whether light can propagate through a nonhomogeneous medium we construct a model which captures, as we believe, some important features of multiscatering phenomena in a random medium. The propagation of light here is treated within geometric optics, i.e., we deal with ideal rays of light propagating in straight lines in a region on a plane. It is assumed that this region consists of infinitely many rectangles with parallel sides neighboring each other in series, and each pair of adjoining rectangles provides passages between them (see Fig. 1). Further, we assume that the ray is reflected off the sides of the rectangles elastically, and is refracted on passage from one rectangle to its neighbor. In addition, the refraction coefficient is assumed to be constant within each rectangle but it may differ from rectangle to rectangle. We model the nonhomogeneity of the medium by supposing that the sizes of the rectangles and (or) the refraction coefficients in them are taken at random. So if the beam of light starts at a point in the region it obviously travels from one rectangle to another experiencing reflection or refraction repeatedly.

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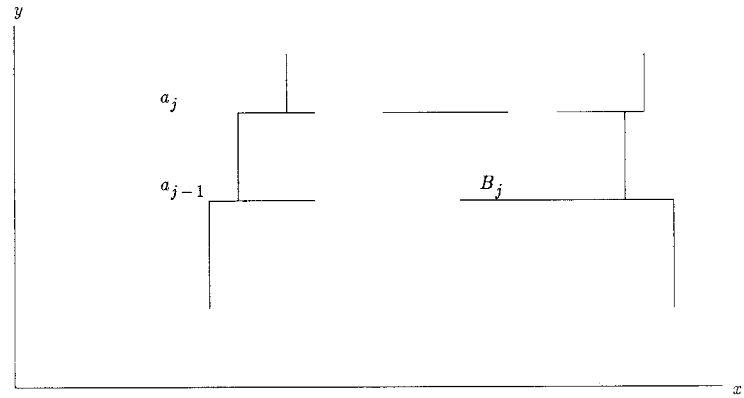

Fig. 1

The main question posed in this article is whether a ray of light propagating in the described region can reach arbitrarily remote regions. One can easily construct simple examples of a system of rectangles with passages and a special ray such that it never leaves some subsystem consisting of a finite number of rectangles. But, as we prove here, with probability 1 it does not happen. More precisely, if the sizes of the rectangles and (or) the coefficients of refraction are random (with a continuous distribution), then with probability 1 the ray does reach arbitrarily remote rectangles.

Let us outline the main points of the proof. The posed problem can be easily reduced to the following problem concerning a system of a finite number of rectangles. Is the trajectory of a ray within such a system dense in it? Notice that if the coefficients of refraction are equal, we have in fact a billiard problem (see the monograph of Kornfeld *et al.*[1]). In particular, the ergodic proerties of some type of billiards and the relevant dynamical system on the torus that corresponds to one rectangle are discussed in this monograph. We extend this construction to build the Riemann surface corresponding to the system of rectangles with passages as well as the appropriate dynamical system on it. We find that the problem about minimality, i.e., density of the trajectories, of this dynamical system can be reduced to a similar problem for the special type of so-called interval exchange transformation (see refs. 1 and 2 for details concerning interval exchange transformations). Using the criterion of minimality[2] and treating the absence of minimality as a sort of a resonance, we find the nonresonance conditions for the parameters of the system in terms of the relevant interval exchange transformations. In fact, these conditions are the countable set of the equations involving arguments which are elementary combinations of the heights of the rectangles, the coefficients of refraction, and the initial angle between the ray and the vertical axis.

This paper is organized as follows. We state our results in the Section 1. Theorem 1.2 is our main result on the propagation of light in the unbounded random medium, as described above. It follows from Theorem 1.3, which states that with probability 1 the trajectory of light within a system of a finite number of rectangles is dense. Theorems 1.2 and 1.3 are proved in Section 3. Theorem 1.6 is a statement about the minimality of a certain class of interval exchange transformations. It forms the basis for the proof of Theorem 1.3. Theorem 1.6 is established in Section 2.

1. STATEMENT OF RESULTS

We start by accurately defining the system of rectangles mentioned in the Introduction. Let $\{a_j, j \in \mathbb{Z}\}$ be a sequence of real numbers on the y axis such that

$$a_i < a_j \quad \text{if } i < j; \qquad \lim_{j \to \pm\infty} a_j = \pm\infty \tag{1.1}$$

The interval $[a_{j-1}, a_j]$ we denote by H_j. Let $\{B_j, j \in \mathbb{Z}\}$ be a sequence of intervals on the x axis such that

$$L(B_{j-1} \cap B_j) > 0 \qquad \text{for} \quad j \in \mathbb{Z} \tag{1.2}$$

where $L(\cdot)$ is the Lebesgue measure on $\mathbb{R}$. The rectangle $B_j \times H_j$ we denote by R_j. Let us denote by R the union of all rectangles R_j. We consider the sides of these rectangles as forming a reflecting surface. We also eliminate a part of this surface in order to have a nontrivial region for light to propagate through.

Definition 1.1. (Passages, reflecting surface.) The passage π_j through the common boundary of neighboring rectangles R_j and R_{j-1} is defined as a finite union of nonempty open intervals $\pi_{j,k}$ included in the interval $R_j \cap R_{j-1}$. If $\tilde{R}$ is the union of all sides of all rectangles R_j and $\tilde{\pi}$ is the union of all passages, then the reflecting surface S is $\tilde{R} \setminus \tilde{\pi}$.

Thus $R \setminus S$ is supposed to be the region within which the ray of light propagates.

We assume that a ray arriving at a reflecting surface is reflected elastically, i.e., if α is the angle between an incident ray and the y axis, then $\pi - \alpha$ is the corresponding angle associated with the reflected ray (see Fig. 2). We also assume that the refraction coefficient $r_j = \sqrt{\varepsilon_j}$ (ε_j is a dielectric constant) is constant within each rectangle R_j, but it can be different in different rectangles. In addition, we suppose that a ray passing between two

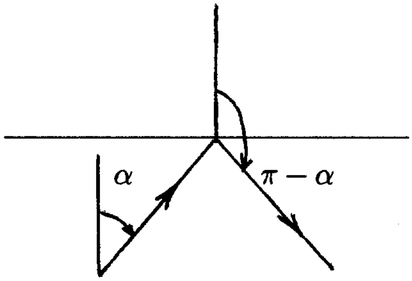

Fig. 2

media I and II with different refraction coefficients r_{I} and r_{II}, respectively, obeys the law (see Fig. 3)

$$r_I \sin \alpha_I = r_{II} \sin \alpha_{II} \tag{1.3}$$

Thus we consider a ray of light starting at a point of the described region and propagating in some direction. It propagates in straight lines within any rectangle. If it reaches the reflecting surface or a passage, it is correspondingly reflected or refracted according to (1.3).

Now we assume some of the parameters describing the region R to be random. Namely, denote by h_j the height of the rectangle R_j, i.e., the length of the interval H_j.

Assumption M. The variables h_j, r_j, $j \in \mathbb{Z}$, are supposed to be random and independent. In addition, (i) for any integer j the random variable $h_j r_j^{-1}$ has a continuous distribution; and (ii) there exist real positive constants r_- and r_+ such that $r_- \leqslant r_j \leqslant r_+$.

Considering the refraction equation (1.3), we may fix r_{I}, r_{II}, and, for instance α_{I}. It may happen that there is no real α_{II} satisfying this equality.

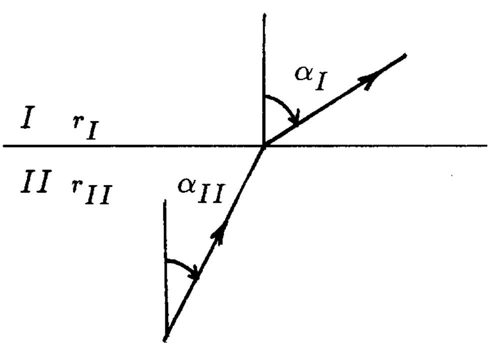

Fig. 3

In optics such a phenomenon is known as full reflection, since in this case the surface between two media behaves like a mirror. In order to prevent this, we impose the following assumption.

Assumption L. Let the ray of light start in the rectangle R_0, making angle α with the y axis. We assume that

$$0 < \alpha < \operatorname{Arcsin} \frac{r_-}{r_+} \tag{1.4}$$

Theorem 1.2. Suppose R to be the region defined above, where each pair of neighboring rectangles has a passage, and a ray of light propagates in R obeying the laws of elastic reflection and refraction (1.3). Assume that horizontal sides B_j of rectangles as well as passages are fixed, whereas their heights and refraction coefficients are random and satisfy Assumption M. Let a ray start at a point in rectangle R_0, with α the angle between it and the y axis, satisfying Assumption L. Then with probability 1 for Lebesgue measure almost all α's from the interval $(0, \operatorname{Arcsin}(r_-/r_+))$ the ray reaches arbitrarily remote parts of R regardless of the starting point in R_0.

Theorem 1.2 is a consequence of the following statement.

Theorem 1.3. Suppose $R^{(n)}$ to be the region formed by the rectangles $R_0, \dots, R_n$, where each pair of neighboring rectangles has a passage, and a ray of light propagates in $R^{(n)}$ obeying the laws of elastic reflection and refraction (1.3). Assume that all other conditions of the Theorem 1.2 are satisfied. Then with probability 1 for Lebesgue measure almost all α's from the interval $(0, \operatorname{Arcsin}(r_-/r_+))$ the trajectory of the ray is a dense set in $R^{(n)}$ regardless to the starting point in R_0.

As we will see, the main problem, in particular Theorem 1.3, can be reduced to a statement about interval exchange transformations. Since it is of a certain interest in itself, we formulate it here. Let us recall the definition of the interval exchange transformation.

Definition 1.4. (Interval exchange transformation.[1,2]) Let $X = [0, L[$ be an interval, $n \geqslant 2$ be an integer, and c_j, $1 \leqslant j \leqslant n$, be positive numbers such that $c_1 + \cdots + c_n = L$. Assume that

$$\beta_0 = 0, \qquad \beta_i = \sum_{j=1}^{i} c_j, \qquad X_i = [\beta_i, \beta_{i-1}[, \qquad 1 \leqslant i \leqslant n$$

Let τ be a permutation of the symbols $\{1, 2, \dots, n\}$. Then

$$c^\tau = \{c_{\tau^{-1}(1)}, c_{\tau^{-1}(2)}, \dots, c_{\tau^{-1}(n)}\}$$

is a vector with positive components, the sum of which equals L. Having c^τ, we can form the corresponding β_i^τ and X_i^τ, $1 \leqslant j \leqslant n$. Now we define $T: X \mapsto X$ by setting

$$Tx = x - \beta_{i-1} + \beta^\tau_{\tau^{-1}(i)-1}$$

for each $x \in X_i$ and each $1 \leqslant i \leqslant n$. Now, T maps each interval X_i isometrically onto the corresponding interval $X^\tau_{\tau^{-1}(i)}$. We call T the interval exchange transformation.

We need also the following fundamental property.

Definition 1.5 (Minimality). An interval exchange transformation T on the interval $[0, L[$ is called minimal if for any $x \in [0, L[$ the set $\{T^m x, m \in \mathbb{Z}\}$ is dense in the interval $[0, L[$.

Let us introduce the function

$$\chi_L(x) = x, \qquad 0 \leqslant x \leqslant L, \qquad \chi_L(x + nL) = \chi_L(x), \qquad n \in \mathbb{Z} \tag{1.5}$$

and the mapping

$$S_\xi(x) = \chi_L(x + \xi), \qquad \xi \in \mathbb{R}, \qquad x \in [0, L[\tag{1.6}$$

Theorem 1.6 (Interval exchange transformations). Let F be an interval exchange transformation on the interval $[0, L[$. Then $S_\xi F$ is an interval exchange transformation on $[0, L[$ for any real ξ, and it is minimal for all ξ except some countable set.

Remark 1.7. Masur[3] and Veech[4] proved that typically (for Lebesgue measure almost all β_i) an interval exchange transformation possesses the minimal property, and even more than that, it is uniquely ergodic. Theorem 1.6 says that the interval exchange transformation can possess the minimal property even if its points of discontinuity $\beta_1, \beta_2, \ldots$ are rationally dependent.

In order to outline the connection between the propagation of the ray of light in a region described in Theorem 1.3 and interval exchange transformations, let us take a segment on the horizontal side of one of the rectangles and let a beam of light start at a point $\mathcal{O}$ on the segment making a fixed angle α with the segment. Assume for simplicity now that all the refraction coefficients are equal. Considering the trajectory of the ray, we note that eventually it crosses again the segment at a point $\mathcal{O}'$ due to Poincaré's theorem (this is true at least for almost all points $\mathcal{O}$). So we have a map by which there corresponds to each point $\mathcal{O}$ the defined point $\mathcal{O}'$ that is connected by a portion of the trajectory of the ray that starts at $\mathcal{O}$ and terminates at $\mathcal{O}'$. Suppose that this portion of the trajectory does not go through the edges of the passages. Then if we move a little bit away from the point $\mathcal{O}$ along the segment, all points of the portion of the trajectory move the same distance in the horizontal direction, in particular, point $\mathcal{O}'$ does. We can proceed in this fashion until the portion of the trajectory crosses an edge of a passage. At this point the mapping $\mathcal{O} \to \mathcal{O}'$ has a break. This observation demonstrates how interval exchange transformations arise in the problem. In addition, as we show in this paper, the density of the trajectory of the ray in the region can be derived from the density of trajectories of the relevant interval exchange transformations.

2. INTERVAL EXCHANGE TRANSFORMATION

For any interval exchange transformation F of the interval $[0, L[$ we denote by

$$D(F) = \{\phi_1, ..., \phi_n\}, \qquad 0 < \phi_1, ..., \phi_n < L \tag{2.1}$$

the set of the points of discontinuity of F.

Definition 2.1 (Irreducibility). An interval exchange transformation T on the interval $[0, L[$ with $D(T) = \{\tau_1, ..., \tau_n\}$ is called irreducible if $T[0, \tau_j[\neq [0, \tau_j[$, $1 \leqslant j \leqslant n$.

To prove Theorem 1.6, we will use the following result.

Proposition 2.2.[2] If T is an irreducible interval exchange transformation on the $[0, L[$, and for any $m \in \mathbb{Z}_+ = \{1, 2, ...\}$,

$$T^m \tau_j \neq \tau_i, \qquad 1 \leqslant i, j \leqslant n \tag{2.2}$$

then T is minimal.

Lemma 2.3. Let F be an interval exchange transformation on the $[0, L[$ and $\xi \in \mathbb{R}$; then:

(i) S_ξ is an interval exchange transformation with the unique point of the discontinuity $L - \xi = S_\xi^{-1}(0)$.

(ii) $S_\xi F$ is an interval exchange transformation, and $D(S_\xi F) \subseteq D(F) \cup \{F^{-1} S_\xi^{-1}(0)\}$.

(iii) If ξ is such that

$$L - \xi \neq F(\phi_j), \qquad 1 \leqslant j \leqslant n \tag{2.3a}$$

or, what is equivalent,

$$S_\xi F(\phi_j) \neq 0, \qquad 1 \leqslant j \leqslant n \tag{2.3b}$$

then $S_\xi F$ is an interval exchange transformation with the following set of points of discontinuity:

$$D(S_\xi F) = D(F) \cup \{F^{-1}S_\xi^{-1}(0)\} = \{\phi_1,..., \phi_n, F^{-1}S_\xi^{-1}(0)\} \tag{2.4}$$

Proof. The statement (i) follows immediately from the definition of S_ξ by (1.6). The equivalence of (2.3a) and (2.3b) follows straightforwardly from (i). To prove (ii), let us notice that if $y \notin \{\phi_1,..., \phi_n, F^{-1}S_\xi^{-1}(0)\}$, then F is continuous at y, and S_ξ is continuous at $F(y)$. Therefore, y is a point of continuity of $S_\xi F$, and $D(S_\xi F) \subseteq \{\phi_1,..., \phi_n, F^{-1}S_\xi^{-1}(0)\}$, which completes the proof of (ii).

Now, if $y = \phi_j$, $1 \leqslant j \leqslant n$, then in accordance with (i) and (2.3b), $F(\phi_j)$ is a point of continuity of S_ξ, and

$$\lim_{h \to \pm 0} S_\xi[F(y+h)] = S_\xi[F(y \pm 0)]$$

Since S_ξ is a one-to-one mapping, and $F(y \pm 0)$ are different, we have $y \in D(S_\xi F)$, and $D(F) \subseteq D(S_\xi F)$. If $y = F^{-1}S_\xi^{-1}(0)$, then, as follows from (2.3b), $y \notin D(F)$ and y is a point of continuity of F. Therefore

$$\lim_{h \to \pm 0} S_\xi F[F^{-1}S_\xi^{-1}(0) + h] = S_\xi[S_\xi^{-1}(0) \pm 0]$$

Since, in accordance with (i), $S_\xi^{-1}(0)$ is a point of discontinuity of S_ξ, we have from the last equality $y = F^{-1}S_\xi^{-1}(0) \subseteq D(S_\xi F)$, which completes the proof of (iii) and the lemma. ∎

Lemma 2.4 (Irreducibility). If $\phi_0 = 0$, F is an interval exchange transformation on the $[0, L[$ with $D(F) = \{\phi_1,..., \phi_n\}$, and ξ is such that

$$S_\xi F(\phi_j) \neq \phi_i, \qquad 0 \leqslant i, j \leqslant n \tag{2.5}$$

then $S_\xi F$ is irreducible.

Proof. Let us notice that the conditions (2.5) contain, in particular, the conditions (2.3) when $i = 0$, and therefore the statement (iii) of Lemma 2.4 is true. Suppose that $S_\xi F$ is reducible, i.e., for some y from $D(S_\xi F)$ we have $S_\xi F[0, y[= [0, y[$. Since $S_\xi F$ is a one-to-one mapping from $[0, L[$ to itself, we have $S_\xi F[y, L[= [y, L[$. Besides, $F^{-1}S_\xi^{-1}(0) \notin [y, L[$, because $S_\xi F[F^{-1}S_\xi^{-1}(0)] = 0$. In particular, $y \neq F^{-1}S_\xi^{-1}(0)$, and since $y \in D(S_\xi F)$, using (2.4), we obtain $y = \phi_j \in D(F)$ for some j, where $1 \leqslant j \leqslant n$. Moreover, all points of discontinuity of $S_\xi F$ on $[y, L[$ are from $D(F)$. Therefore there exists such $z \in [y, L[$ that $S_\xi Fz = y$. If we assume that $z \notin D(F)$, then $z \in]y, L[$, and z is a point of continuity of $S_\xi F$. From this we would conclude that $S_\xi Fz \in]y, L[$, which contradicts $S_\xi Fz = y$.

Thus $z=\phi_i \in D(F)$ for some i, where $1 \leqslant i \leqslant n$, and we have $S_\xi F(\phi_j)=\phi_i$. But the last equality is in contradiction to (2.5). Thus, the lemma is true. ▮

Let us introduce the following functions:

$$s_1(\xi, x)=S_\xi F(x)=\chi_L(\xi+F(x)), \qquad \xi \in \mathbb{R}, \qquad x \in [0, L[\tag{2.6}$$

$$s_m(\xi, x)=s_1(\xi, s_{m-1}(\xi, x)), \qquad m=2, 3,... \tag{2.7}$$

Lemma 2.5. We have the following statements:

(i) $s_m(\cdot, \cdot)$: $\mathbb{R} \times [0, L[\mapsto [0, L[$, $s_m(\xi, x)=(S_\xi F)^m x$, $m \in \mathbb{Z}_+$.

(ii) The functions $s_m(\cdot, \cdot)$ are periodic functions of ξ, i.e.,

$$s_m(\xi+kL, x)=s_m(\xi, x), \qquad k \in \mathbb{Z}, \qquad m \in \mathbb{Z}_+$$

(iii) If x is any fixed number from $[0, L[$, and $\tilde{s}_m(\xi)=s_m(\xi, x)$, $\xi \in [0, L[$, then

$$\tilde{s}_m(\xi)=s_1(\xi, \tilde{s}_{m-1}(\xi)), \qquad \tilde{s}_m(\cdot)\text{: } [0, L[\mapsto [0, L[\tag{2.8}$$

Proof. All statements of this lemma are straightforward consequences of the definitions (1.5), (1.6), (2.6), and (2.7). ▮

As follows from the constructions, the functions $\tilde{s}_m$ are piecewise linear. In order to prove it and find their properties, we introduce the following classes of functions.

Definition 2.6. Let $\mathscr{F}_m$, $m \in \mathbb{Z}_+$, be a class of functions $s(\cdot)$: $[0, L[\mapsto [0, L[$ each possessing the following property: there exists a nonnegative integer $k=k(s(\cdot))$, and two sets of real numbers $\{l_j\}_0^{k+1}$, $\{c_j\}_0^k$ depending on $s(\cdot)$, such that

$$0=l_0<l_1<\cdots<l_k<l_{k+1}=L \tag{2.9}$$

$$s(\xi)=m\xi+c_j, \qquad \xi \in [l_j, l_{j+1}[, \qquad 0 \leqslant j \leqslant k \tag{2.10}$$

The set $D(s)=\{l_1,..., l_k\}$ is supposed to be a set containing points of discontinuity of $s(\cdot)$.

Lemma 2.7. Let $\mathscr{F}_m$, $m \in \mathbb{Z}_+$, be the classes of functions defined above. Then:

(i) If F is an interval exchange transformation on $[0, L[$ and $s(\cdot) \in \mathscr{F}_m$, then $F \in \mathscr{F}_1$ and $F(s(\cdot)) \in \mathscr{F}_m$.

(ii) If $s(\cdot)\in\mathscr{F}_m$, then $s_1(\cdot, s(\cdot))\in\mathscr{F}_{m+1}$.

(iii) $\tilde{s}_m(\cdot)\in\mathscr{F}_m$, $m\in\mathbb{Z}_+$.

(iv) If $y\in[0, L[$, then the number of solutions of the equation

$$\tilde{s}_m(\xi)=y \tag{2.11}$$

is finite.

Proof. The fact that belongs to $\mathscr{F}_1$ obviously follows from Definition 1.4. Let $D(Fs)$ be a set of the points of discontinuity of the mapping $F(s(\cdot))$. Then $D(Fs)\subseteq D(s)\cup s^{-1}(D(F))$. Since the last set is obviously finite, we have the associated finite partition of $[0, L[$ into smaller half-open intervals J_k, and $s(\cdot)$ as well as $F(s(\cdot))$ are continuous within each such J_k. Having this, we easily obtain the representation (2.10) for $F(s(\cdot))$ on each J_k, which completes the proof of (i). The statement (ii) follows from (i) and (2.6). The statement (iii) for $\tilde{s}_1$ follows immediately from (2.6). The validity of this statement for arbitrary natural number m can be obtained by induction from (2.8) and the statement (ii) of this lemma. Finally, (iv) is a straightforward consequence of (iii). This completes the proof of the lemma. ▮

Definition 2.8. Let F be the interval exchange transformation on $[0, L[$ with $D(F)=\{\phi_1,\dots,\phi_n\}$, $0<\phi_1,\dots,\phi_n<L$, and $\phi_0=0$. We call a number $\xi\in\mathbb{R}$ resonant to F if there exist a natural numer m and nonnegative integers i, j, $0\leqslant i, j\leqslant n$, such that $(S_\xi F)^m\,\phi_i=\phi_j$. If ξ is not resonant, we call it nonresonant. The set $R_F\subset\mathbb{R}$ of numbers resonant to F we call the resonant set, and its complement $N_F=\mathbb{R}\backslash R_F$, i.e., the set of nonresonant numbers, we call the nonresonant set.

Lemma 2.9. Following the notations of the previous definition, we have:

(i) If x and y are any fixed numbers from $[0, L[$ and m is any natural number, then the number of ξ which satisfy the equation

$$(S_\xi F)^m\,x=y \tag{2.12}$$

is finite.

(ii) The resonant set R_F is countable.

(iii) The nonresonant set N_F can be defined as the set of ξ that satisfy the following nonresonant conditions:

$$(S_\xi F)^m\,\phi_i\neq\phi_j,\qquad m\in\mathbb{Z}_+,\qquad 0\leqslant i, j\leqslant n \tag{2.13}$$

Proof. The statement (i) follows immediately from Lemma 2.5 and Lemma 2.7(iv). The fact that R_F is countable follows from its definition and (i). The statement (iii) follows straightforwardly from Definition 2.8. ▌

Proof of Theorem 1.6. Let us show that if $\xi \in N_F = \mathbb{R} \setminus R_F$, then $S_\xi F$ satisfies the conditions of Proposition 2.2. First, in accordance with Lemma 2.9(ii), N_F differs from $\mathbb{R}$ on a countable set only. Second, as follows from Lemma 2.9, if $\xi \in N_F$, then the nonresonant conditions (2.13) are true. Since these conditions obviously cover the conditions (2.5) of Lemma 2.4, $S_\xi F$ is an irreducible interval exchange transformation. Besides, the conditions (2.13) cover also the conditions of Lemma 2.3, and therefore the representation (2.4) is true. Using this representation and recalling that $\phi_0 = 0$, we easily find that for $T = S_\xi F$ the conditions (2.13) yield the inequalities (2.2). This completes the proof of Theorem 1.6. ▌

To provide the constructions establishing Theorems 1.2 and 1.3 by necessary elements, we need some more concepts.

Definition 2.10 (Induced map). Let T be an automorphism of the Lebesgue space (X, m), and suppose that Y is a measurable subset of X, $m(Y) > 0$. For all $y \in Y$ we set $k(y) = \min\{k: k \in B\}$ if $B \neq \emptyset$, and $k(y) = +\infty$ if $B = \emptyset$, where $B = \{k \in \mathbb{Z}_+: T^k y \in Y\}$. If $\tilde{Y} = \{y: k(y) < +\infty\}$, then by Poincaré's theorem $m(Y) = m(\tilde{Y})$. The mapping $T'_Y: y \mapsto T^{k(y)}y$, $y \in \tilde{Y}$, is called the induced map generated by T on Y.

Definition 2.11 (Multi-interval exchange transformation). Let I be a union of a finite number of disjoint half-open intervals $I_k = [a_k, b_k[$. Let us call a one-to-one mapping from I to I a multi-interval exchange transformation if there exist a partition of each half-open interval into a finite number of half-open intervals $I_{k,j}$ and corresponding constants $c_{k,j}$ such that

$$Tx = x + c_{k,j}, \qquad x \in I_{k,j} \tag{2.14}$$

The defined multi-interval exchange transformations arise naturally as induced maps generated by interval exchange transformations T, defined on the interval $[0, L[$, and I, which is a union of a finite number of disjoint half-open intervals I_k included in $[0, L[$. It is the subject of the following lemma, which generalizes the corresponding result when I is an half-interval.[2]

Lemma 2.12 (Multi-interval exchange transformation). Using the notations of Definition 2.11, suppose that F is an interval exchange transformation on $[0, L[$ and $I \subseteq [0, L[$. Then the induced map F'_I is a multi-interval exchange transformation. The set $D(F'_I)$ of the points of discontinuity of F_I' can be found as follows. Let us assign to each x from $[0, L[$ the nonnegative number $l(x)=\min\{l: l\in B\}$ if $B\neq\varnothing$, and $l(x)=+\infty$ if $B=\varnothing$, where $B=\{l>0: T^{-l}x\in I\}$. Then each point of discontinuity of F_I' is of the form $T^{-l(x)}x$, where x is either a point of discontinuity of F or an endpoint of I with $l(x)<+\infty$.

Proof. Let us consider the partition of the set I into smaller half-open intervals, associated with the points $T^{-l(x)}x$, described in the statement of the lemma. Let $[a, b[$ be any interval of this partition. In other words, $[a, b[$ is a subset of some I_k, and each of a or b is either an endpoint of I_k or has the form $T^{-l(x)}x$, mentioned above. We should prove that F_I' is continuous on the $[a, b[$. In order to do so, assume that $y\in\,]a, b[$, and that for some natural number m

$$F^m y\in I, \qquad F^l y\notin I, \quad 1\leqslant l\leqslant m-1 \tag{2.15}$$

We state that

$$F^l y\notin D(F)=\{\phi_1,\ldots,\phi_n\}, \qquad 1\leqslant l\leqslant m-1 \tag{2.16}$$

and

$$F^m y\in \mathring{I} \tag{2.17}$$

where $\mathring{I}$ is the interior of I. Indeed, assume that for some appropriate l, $F^l y=\phi\in D(F)$, i.e., $y=F^{-l}\phi$. Then $F^{-l(\phi)}\phi\in I$, where $1\leqslant l(\phi)\leqslant l$, and $F^{l-l(\phi)}y=F^{-l(\phi)}\phi\in I$. Combining the last relation with (2.15), we must set $l(\phi)=l$. But in this case $y=F^{-l(\phi)}\phi$ is in $]a, b[$, which contradicts the construction of the interval $[a, b[$. Therefore, if (2.15) is true, then (2.16) is true.

Now supposing (2.15), assume that (2.17) is not true, i.e., $F^m y=e$, where e is an endpoint of I. Then, $y=F^{-m}e$ and therefore $F^{-l(e)}e\in I$, $1\leqslant l(e)\leqslant m$. In addition, $F^{m-l(e)}y=F^{-l(e)}e\in I$, which together with (2.15) implies $l(e)=m$, and $y=F^{-l(e)}e$. But the last relation contradicts the construction of the interval $[a, b[$. Thus, if (2.15) is true, then both (2.16) and (2.17) are true.

The validity of (2.16) and (2.17) in turn implies that F^l, $1\leqslant l\leqslant m$, are continuous mappings within some open interval $O_y\ni y$, and

$$F_I' x=F^m x, \qquad x\in O_y; \qquad F^m O_y \text{ is an interval in } I \tag{2.18}$$

Now, let us prove that the equality in (2.18) holds on the whole interval $]a, b[$. Recalling Definition 2.10, we suppose $T=F$, $X=[0, L[$, $m=L$ is Lebesgue measure on $[0, L[$, and $Y=I$. In addition, we set

$$m_0=\operatorname*{essmin}_{y\in\,]a,b[} k(y) \tag{2.19}$$

i.e.,

$$k(y) \geqslant m_0, \qquad \text{a.e. in }]a, b[\tag{2.20}$$

Then we obviously have

$$L\{y \in]a, b[: k(y) = m_0\} > 0 \tag{2.21}$$

Combining (2.18) and (2.20), we obtain

$$k(y) \geqslant m_0, \qquad y \in]a, b[\tag{2.22}$$

We can notice from Definition 2.10 that $m_0 \geqslant 1$. Now we will prove that F^{m_0} is a continuous mapping on $[a, b[$. From (2.22) we have

$$F^l y \notin I, \qquad 1 \leqslant l \leqslant m_0 - 1, \quad y \in]a, b[\tag{2.23}$$

Let us show that

$$F^l y \notin D(F), \qquad 0 \leqslant l \leqslant m_0 - 1, \quad y \in]a, b[\tag{2.24}$$

If $m_0 = 1$, it is obviously follows from the construction of the interval $[a, b[$. If $m_0 > 1$, the assumption that $F^l y = \phi \in D(F)$ for appropriate $l \geqslant 1$ leads to a contradiction. Indeed, if the last is true, we have $]a, b[\ni y = F^{-l}\phi$, and then $F^{-l(\phi)}\phi \in I$, $1 \leqslant l(\phi) \leqslant l$, and $F^{l-l(\phi)} y \in I$. Combining the last relation with (2.23), we must set $l(\phi) = l$, which leads to $y = F^{-l(\phi)}\phi \in I$, which contradicts the construction of $[a, b[$. Therefore, (2.24) is true. This implies in turn that F^{m_0} is continuous on $]a, b[$. Therefore $F^{m_0}]a, b[$ is an interval. Moreover, from (2.21) we have $F^{m_0}]a, b[\cap I \neq \varnothing$. Let us prove that

$$F^{m_0}]a, b[\subseteq \mathring{I} \tag{2.25}$$

To establish (2.25), it is sufficient to prove for any $y \in]a, b[$ that $F^{m_0} y$ is not an endpoint of I. This proof is literally the same as the proof of (2.24), in particular, it exploits (2.23). Thus (2.25) is true. Since F^{m_0} is continuous on the right, we have from (2.25) $F^{m_0}]a, b[\subseteq I$, and therefore the interval $F^{m_0}]a, b[\subseteq I_j$ for some j. From this and inequality (2.20) we can infer that $k(y) = m_0$, $y \in [a, b[$, and then conclude that

$$F_I' \mid_{[a,b[} = F^{m_0} \mid_{[a,b[} \tag{2.26}$$

Therefore $F_I'[a, b[$ is an interval included in some I_j. The last remark, (2.26), and the fact that the intervals $[a, b[$ form the partition of I imply easily the lemma. ∎

3. DYNAMICAL SYSTEMS INTERPRETATION AND PROOF OF MAIN RESULTS

We begin with a consideration of Theorem 1.3. The rigorous proof is rather lengthy and requires, first, the replacement of rectangles R_j by the appropriate tori (see Proposition 3.1 below) and, second, the construction of the relevant Riemann surface. This is why we give first an informal proof, cutting out some technical details.

Informal Arguments. For simplicity we now suppose that all the refraction coefficients are equal, so we have a billiard dynamical system.[1] Let us introduce notation as is shown in Fig. 4. Suppose a ray starts at a point $\mathcal{O}$ on the upper side of the rectangle R_n. The statement of Theorem 1.3 is that for Lebesgue measure almost all $h_0, h_1, \ldots, h_n$, α and any point $\mathcal{O} \in [\mathcal{A}_n, \mathcal{B}_n]$ the trajectory of the ray is a dense set in the region $R^{(n)}$. Let us fix now the horizontal sides of the rectangles R_j and the passages π_j (see Definition 1.1) through them, $0 \leqslant j \leqslant n$. We shall prove the desired statement by induction on n. To do so, we assume that Theorem 1.3 is true for the system of rectangles $R_0, \ldots, R_{n-1}$. Then let us fix $h_0, h_1, \ldots, h_{n-1}$ for which the statement of Theorem 1.3 holds. We want to show that for Lebesgue measure almost all h_n and α the trajectory of the ray is dense in R_n. To do this, it is sufficient to check whether the trajectory is dense in the interval $[\mathcal{A}_n, \mathcal{B}_n]$. Let $\mathcal{O} \in [\mathcal{A}_n, \mathcal{B}_n]$ (see Fig. 4). Suppose the ray starts at $\mathcal{O}$. Let $\mathcal{O}'$ denote the point of the first return on the side $[\mathcal{A}_n, \mathcal{B}_n]$ (this time is finite at least for almost all points $\mathcal{O}$). Denote by F_n the corre-

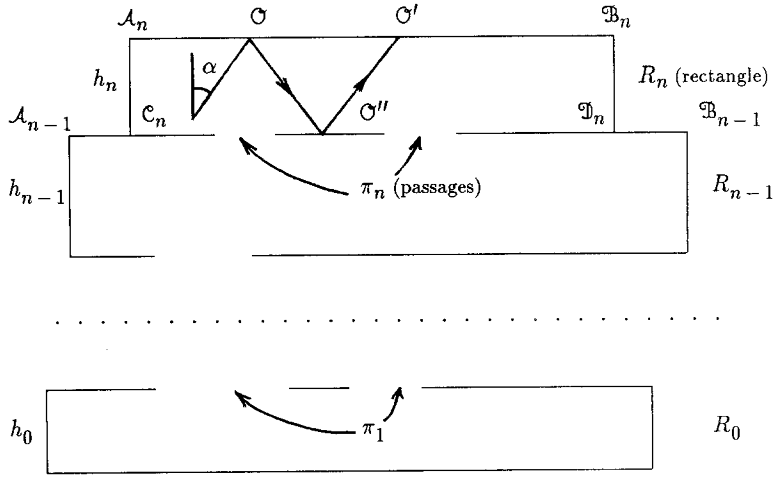

Fig. 4

sponding automorphism on $[\mathscr{A}_n, \mathscr{B}_n]$, i.e., $F_n\colon \mathcal{O} \mapsto \mathcal{O}'$. We include in the induction assumption the statement that the relevant F_{n-1} is an interval exchange transformation on the interval $[\mathscr{A}_{n-1}, \mathscr{B}_{n-1}]$ (see Fig. 4 and Definition 1.4). Now let us show that F_n possesses the following properties:

(i) F_n is an interval exchange transformation (for a finite number of intervals!) on $[\mathscr{A}_n, \mathscr{B}_n]$, and therefore it is defined for all points $\mathcal{O}$.

(ii) $\forall \mathcal{O} \in [\mathscr{A}_n, \mathscr{B}_n]$ the set $\{F_n^m \mathcal{O},\ m = 1, 2,...\}$ is dense in $[\mathscr{A}_n, \mathscr{B}_n]$.

To prove (i) and (ii), let us reason as follows. Starting at a point $\mathcal{O}$, the ray crosses the lower side $[\mathscr{C}_n, \mathscr{D}_n]$ at a point $\mathcal{O}''$. There are two possibilities here: (a) the point $\mathcal{O}''$ does not belong to any of the passages from π_n between R_n and R_{n-1}; (b) $\mathcal{O}''$ belongs to a passage from π_n. In the case (a) the ray reflects off the lower side and $\mathcal{O}' = \mathcal{O} + 2h_n \sin\alpha$ (rigorously speaking, the last equality can be completely justified after the replacement of the rectangles by the corresponding tori; see Proposition 3.1 below). In other words, for those points, F_n acts just as the shift by $2h_n \sin\alpha$. In case (b) the ray crosses one of the passages and moves into the rectangle R_{n-1}. Then, traveling within the rectangles $R_{n-1},..., R_0$, the ray eventually returns to one of the passages between R_n and R_{n-1}. This must happen by the inductive hypothesis. Le now F'_{n-1} be the induced automorphism generated by the mapping F_{n-1} on the interval $[\mathscr{A}_{n-1}, \mathscr{B}_{n-1}]$ and the set π_n which is a finite set of intervals (passages). Then we define the automorphism $\tilde{F}_{n-1}\colon [\mathscr{C}_n, \mathscr{D}_n] \mapsto [\mathscr{C}_n, \mathscr{D}_n]$ by the formula

$$\tilde{F}_{n-1} x = \begin{cases} F'_{n-1} x, & \text{if } \ x \in \pi_n \\ x, & \text{if } \ x \in [\mathscr{C}_n, \mathscr{D}_n] \backslash \pi_n \end{cases}$$

Now we have the following recursive relationship:

$$F_n = S_\xi \circ \tilde{F}_{n-1} \circ S_\xi, \qquad \xi = 2h_n \sin\alpha$$

where S_ξ is the shift by ξ on the intervals $[\mathscr{A}_n, \mathscr{B}_n]$ or $[\mathscr{C}_n, \mathscr{D}_n]$ (as we noticed before, these intervals should be replaced by the relevant circles). From the relationships obtained we may conclude that if F_{n-1} is an interval exchange transformation, then F_n is an interval transformation, too. Then the problem of whether the map F_n has dense trajectories is equivalent to the same problem for the map $S_\xi \circ F_n \circ S_\xi^{-1} = S_\xi^2 \circ \tilde{F}_{n-1}$. Now may use Theorem 1.6, since $\xi = 2h_n \sin\alpha$ and we can choose h_n as we wish. This completes our informal arguments.

Now let us consider the rigorous proof of Theorem 1.3. If the refraction coefficients are equal, actually we have a problem about a billiard dynamical system.[1] The construction of a billiard system might be naturally extended to cover the refraction phenomenon. In the special case of the system of rectangles that we deal with here, we can use the following representation of trajectories of the ray as a projection of relevant trajectories on the associated tori.

Proposition 3.1 (Torus[1]). Let $A = [0, b] \times [0, h]$ be a rectangle in the xy plane, considered as a phase space for the billiard system, and $\hat{A} = [-b, b] \times [-h, h]$ is a torus, i.e., we identify horizontal and vertical sides, respectively. Consider the trajectory $q(t)$, $t \geqslant 0$, on A that starts at $q(0) = q_0$ in the direction of a unit vector w. If the operator $J: \mathbb{R}^2 \mapsto \mathbb{R}^2$ is defined by

$$J(x, y) = (|x|, |y|), \qquad (x, y) \in \mathbb{R}^2 \tag{3.1}$$

and $Q(t) = q_0 + wt$, $t \geqslant 0$, is a trajectory on the rotus $\hat{A}$, then (see Fig. 5)

$$q(t) = JQ(t), \qquad t \geqslant 0 \tag{3.2}$$

Now our plan is as follows:

(i) To represent each rectangle R_j as a torus $\hat{R}_j$, $0 \leqslant j \leqslant n$, considered as a sheet of a Riemann surface (namely, if we have in the local coordinates $R_j = [0, b_j] \times [0, h_j]$, then $\hat{R}_j$ is the rectangle $[-b_j, b_j] \times [-h_j, h_j]$ in which we identify the vertical and horizontal sides, respectively).

(ii) To construct the Riemann surface $\hat{R}^{(n)}$ by making cuts on each torus $\hat{R}_j$ along the passages $\pi_{j,k}$ and $\pi_{j+1,k}$ (see Definition 1.1), and identifying the appropriate edges of these cuts in such a way that if a point within $\hat{R}_j$ moves toward the cut along $\pi_{j,k}$ or $\pi_{j+1,k}$, then it passes through the cut to the torus $\hat{R}_{j-1}$ or $\hat{R}_{j+1}$, respectively.

(iii) To assign the vector field v on the Riemann surface $\hat{R}^{(n)}$ using the relations (1.3), and (1.4), and having the initial angle α in the rectangle R_0 fixed.

Thus, using Proposition 3.1, we replace the rectangles R_j by the corresponding tori $\hat{R}_j$, denoting by $L^u(j)$ and $L^l(j)$, respectively, the lines on the

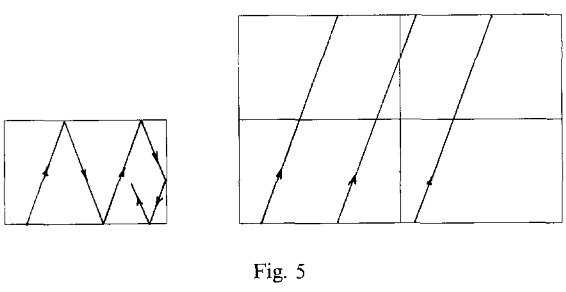

Fig. 5

torus $\hat{R}_j$ associated with the upper and the lower sides of the rectangle R_j. More precisely, in the local coordinates used in Proposition 3.1, $L^l(j) = \{y=0\}$, $L^u(j) = \{y=h_j\}$. Using the same local coordinates, we define the following sets of cuts on the tori corresponding to the passages $\pi_{j,k}$ (see Definition 1.1)

$$C^l(j) = \{\pm\pi_{j,k}\}, \qquad C^u(j) = \{\pm\pi_{j+1,k}\} \tag{3.3a}$$

$$C^l(0) = C^u(n) = \varnothing \tag{3.3b}$$

Remark 3.2. Each set $C^l(j)$ or $C^u(j)$ consists of a finite union of intervals. To keep the main statement true for any starting point, we will assume below that each such interval includes the left endpoint and does not include the right one. Except for this, we leave these sets without change. As a matter of fact, such an assignment of endpoints means a certain choice of the reflection at the endpoints of the passages.

Now we construct the Riemann surface $\hat{R}^{(n)}$ as follows. Having the sheets $\hat{R}_j$ $(0 \leqslant j \leqslant n)$, we cut each $\hat{R}_j$ along the sets $C^l(j)$ and $C^u(j)$. Since each cut generates two edges ($+$ or $-$), we have two copies of each set $C^l(j)$ and $C^u(j)$, namely $C^l_\pm(j)$ and $C^u_\pm(j)$, respectively. The sign $+$ or $-$ of an edge is assigned in a uniform manner according to the direction of the y axis (see Fig. 6). Noticing from (3.3) that the sets $C^l(j+1)$ and $C^u(j)$ are generated by the same passages, we may join the sheets by the following identification:

$$C^l_\pm(j+1) \equiv C^u_\mp(j) \tag{3.4}$$

Equation (3.4) means that we identify the intervals generated by the same passages $\pm\pi_{j,k}$. This completes the construction of the Riemann surface $\hat{R}^{(n)}$.

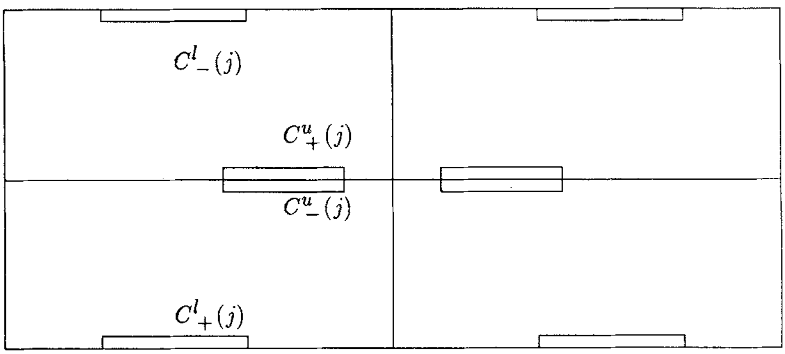

Fig. 6

Let us assign the vector field v on the our Riemann surface $\hat{R}^{(n)}$ as follows:

$$v(Q) = (r_j \sin \alpha_j, r_j \cos \alpha_j), \qquad Q \in \hat{R}_j \tag{3.5}$$

$$r_j \sin \alpha_j = r_0 \sin \alpha_0, \qquad \alpha_0 = \alpha, \quad 0 < \alpha_j < \operatorname{Arcsin} \frac{r_-}{r_+}, \quad 0 \leqslant j \leqslant n \tag{3.6}$$

Equalities (3.5)–(3.6) define correctly the piecewise constant vector field v any where on the Riemann surface except on former edges of the cuts. Having v defined on this set, we can define v on the whole surface by

$$v(Q) = v(x, y) = \lim_{\eta \to +0} \lim_{\delta \to +0} v(x + \eta, y + \delta) \tag{3.7}$$

where (x, y) is the local coordinate of Q.

That is, the trajectory $Q(t)$ on the surface $\hat{R}^{(n)}$ is defined by

$$\frac{d}{dt} Q = v(Q), \qquad Q(0) = Q_0 \tag{3.8}$$

If we introduce the mapping V^t: $Q_0 \mapsto Q(t)$, we can write (3.8) in the form

$$Q(t) = V^t Q_0, \qquad Q_0 \in \hat{R}^{(n)}, \qquad t \in \mathbb{R} \tag{3.9}$$

Lemma 3.3. If $\hat{l}_n$ is Lebesgue measure on the Riemann surface $\hat{R}^{(n)}$, then V^t, $t \in \mathbb{R}$, is a group of automorphisms on $R^{(n)}$ and $(V^t, \hat{R}^{(n)}, \hat{l}_n)$ forms a dynamical system. In addition:

(i) V^t, $t \in \mathbb{R}$, preserves the measure $\hat{l}_n$.

(ii) The trajectory $q(t)$ of the ray of light on $R^{(n)}$ can be represented by

$$q(t) = JQ(t), \qquad t \in \mathbb{R} \tag{3.10}$$

where the operator J acts on each sheet $\hat{R}_j$ as defined in Proposition 3.1.

Proof. The first statement follows from (3.8), (3.9), and the definition of a dynamical system.[1] The statements (i) and (ii) follow from (3.5)–(3.9) and Assumption L. ∎

As follows from the constructions (3.5)–(3.9) of the vector field v and the group V^t, a point Q moves under the action of V^t rectilinearly within any sheet of the Riemann surface until it gets to the intervals of former cuts. At these intervals (passages) the point passes to another sheet (rectangle) with a possible change of the direction (as a result of refraction).

Since it is on these intervals where the action of the group changes its form, we want to focus our attention on these phases of the evolution. Thus, let us construct an operator G_n that is associated with V^t and maps the lines, along which the sheets are connected, to themselves.

To do so, we consider the lines $L^u(j)$, $L^l(j)$ defined above, and their images $\hat{L}^u(j)$, $\hat{L}^l(j)$ on the Riemann surface. We should specify what edge of the cut belongs to each line. We do this in the following manner:

$$\hat{L}^u(j) = N^u(j) \cup C^u_-(j), \qquad N^u(j) = L^u(j) \backslash C^u(j) \tag{3.11}$$

$$\hat{L}^l(j) = N^l(j) \cup C^l_-(j), \qquad N^l(j) = L^l(j) \backslash C^l(j) \tag{3.12}$$

The motivation for this choice is to join to the line with argument j the edge that goes to a line with the argument $j-1$ or $j+1$. Denote by $\mathscr{L}_n$ the union of all the sets $\hat{L}^l(j)$, $\hat{L}^u(j)$. Now we define the G_n as follows:

$$G_n Q = V^{t_1} Q, \qquad t_1 = \min\{t > 0\colon V^t Q \in \mathscr{L}_n\}, \qquad Q \in \mathscr{L}_n \tag{3.13}$$

This construction of G_n is analogous to the construction of the relevant mapping for billiards.[1] Let us list the properties of the operator G_n.

Lemma 3.4. If $L(\cdot)$ is Lebesgue measure on the set of lines $\mathscr{L}_n$, then G_n is an automorphism on $\mathscr{L}_n$ preserving the measure $L(\cdot)$. In addition, it maps points of a line to the points of closest lines

$$G_n(N^u(j)) \subset \hat{L}^l(j), \qquad G_n(C^u_-(j)) \subset \hat{L}^u(j+1) \tag{3.14}$$

$$G_n(N^l(j)) \subset \hat{L}^u(j), \qquad G_n(C^l_-(j)) \subset \hat{L}^u(j-1) \tag{3.15}$$

Proof. These statements follow straightforwardly from (3.11), (3.12), and the definitions of the group V^t and the mapping G_n. ∎

It is clear that the action of the G_n could be represented somehow by the shift operators S_ξ defined in (1.6). Since the intervals $\hat{L}^l(n)$ and $\hat{L}^u(n)$ have the same length $2b_n$ (b_n is the length of the horizontal side of the rectangle R_n), we may introduce trivially the isomorphism $I(n)$ from $\hat{L}^u(n)$ to $\hat{L}^l(n)$ (see Fig. 7). By the same reason, in view of (3.3), we may introduce in a natural way also the isomorphism $J_-(n)$ from $C^u_-(n-1)$ to $C^l_-(n)$ (see Fig. 7).

Let us set

$$\xi_n = h_n r_n^{-1} r_0 \sin\alpha \tag{3.16}$$

where h_n is the height of the rectangle R_n.

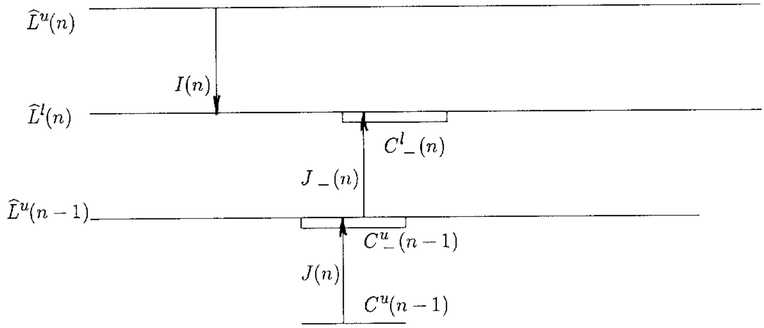

Fig. 7

Lemma 3.5. If S_{ξ_n} is the shift operator defined by (1.6) on the interval $\hat{L}^u(n)$ of the length $L = 2b_n$, then

$$G_n(\hat{L}^u(n)) = \hat{L}^l(n), \qquad G_n \mid_{\hat{L}^u(n)} = I(n) \circ S_{\xi_n} \tag{3.17}$$

$$G_n \mid_{\hat{N}^l(n)} = S_{\xi_n} \circ I(n)^{-1} \tag{3.18}$$

$$G_n \mid_{C^u_-(n-1)} = S_{\xi_n} \circ I(n)^{-1} \circ J_-(n) \tag{3.19}$$

Proof. The first equality in (3.17) a simple consequence of Lemma 3.4 and (3.3b). The second one as well as the relations (3.18) and (3.19) can be obtained by elementary computation from the definitions of v by (3.5)–(3.6), G_n by (3.13), and (3.16). ▮

Now we introduce the following induced map (see Definition 2.10):

$$F_n = (G_n)'_{\hat{L}^u(n)}\colon\ \hat{L}^u(n) \mapsto \hat{L}^u(n) \tag{3.20}$$

Our next goal is the following: we want to show that F_n is an interval exchange transformation by finding an appropriate representation for F_n. Then we can use the results of the previous section to prove that F_n is minimal with probability 1, and therefore any trajectory $F_n^m x$, $x \in \hat{L}^u(n)$, $m \in \mathbb{Z}_+$, is dense in $\hat{L}^u(n)$.

To realize this program, let us find the recurrence formula for F_n as a function of n, and then apply induction. Having the region $R^{(n)}$, we consider the region $R^{(n-1)}$ formed by the same rectangles $R_0,..., R_{n-1}$ as well as by the same passages $C^l(j)$, $C^u(j)$ except for $C^u(n-1)$, which is eliminated, i.e., supposed to be empty. Applying all previous construction to the case of $R^{(n-1)}$, we will have the corresponding Riemann surface $\hat{R}^{(n-1)}$ and mappings G_{n-1} and F_{n-1}. Now let

$$F'_{n-1} = (F_{n-1})'_{C^u(n-1)} \tag{3.21}$$

be the induced map acting on $C^u(n-1)$. Denoting by $J(n)$ the natural isomorphism from $C^u(n-1)$ to $C^u_-(n-1)$ (see Fig. 7), we introduce also the following mapping, which acts on $\hat{L}^l(n)$:

$$F''_{n-1}(x) = \begin{cases} J_-(n) \circ J(n) \circ F'_{n-1} \circ J(n)^{-1} \circ J_-(n)^{-1}\, x & \text{if } x \in C^l_-(n) \\ x & \text{otherwise} \end{cases} \tag{3.22}$$

Remark 3.6. Let us consider the meaning of the introduced mappings F_n, F'_{n-1}, and F''_{n-1}. Under the sequential action of the automorphism G_n, i.e., G_n, G_n^2,..., a point $Q \in \mathscr{L}_n$ jumps from one line in the set $\mathscr{L}_n$ to another one. As follows from Definition 2.10, F_n maps a point on $\hat{L}^u(n)$ to the point of the first return to $\hat{L}^u(n)$. Therefore, considering the sequence $G_n^m Q$, $m = 1, 2,...$, we can observe two options.

The first one is if $G_n Q$ is in $N^l(n)$, and then, in view of (3.15) and (3.18), $G_n^2 Q \in \hat{L}^u(n)$. In other words, this is the case when the point $Q \in \hat{L}^u(n)$ returns to the same line without getting out from the nth sheet.

The second option occurs when $G_n^2 Q \in C^l_-(n)$, and in accordance with (3.15), $G_n^2 Q$ is in the $(n-1)$th sheet. This means that the point $G_n Q$ is within the system $R^{(n-1)}$ up to the identification of the edges. The operator F'_{n-1} is designed to map the "coming in" point to the "coming out" one. In addition, F'_{n-1} as a induced map acts as the minimal positive power of G_{n-1} that returns the point Q from $\hat{L}^u(n-1)$ back. Therefore in this case abusing the notation and using (3.14), we might write $F_n Q = G_n F'_{n-1} G_n Q$. The operator F''_{n-1} is designed to join both cases, taking into account appropriate isomorphisms between cuts and their edges. ▮

Lemma 3.7. The following recurrence formula holds:

$$F_n = S_{\xi_n} \circ I(n)^{-1} \circ F''_{n-1} \circ I(n) \circ S_{\xi_n}, \qquad n \geqslant 2 \tag{3.23}$$

Proof. As was explained in Remark 3.6, we will consider two cases. In the first case we may state that

$$F_n Q = G_n^2 Q, \qquad Q \in N^l(n) \tag{3.24}$$

In the second case, using the definition of the induced map, we obtain

$$F_n Q = G_n \circ J(n) \circ F'_{n-1} \circ J(n)^{-1} \circ J_-(n)^{-1}\, G_n^2 Q, \qquad Q \in C^l(n) \tag{3.25}$$

Combining (3.24) and (3.25) with (3.17)–(3.19) and (3.22), we get (3.23), which completes the proof of the lemma. ▮

Lemma 3.8. F_n is an interval exchange transformation.

Proof. For $n=1$ this statement is trivial since

$$F_1 = S_{\xi_n} \circ S_{\xi_n} \tag{3.26}$$

Assume that the statement is true for $n-1$, i.e., F_{n-1} is an interval exchange transformation. Then, since $C^u(n-1)$ by the definition is the union of a finite number of half-open intervals, F_{n-1}, in accordance with Lemma 2.12, is a multi-interval exchange transformation on $C^u(n-1)$. From this and the recurrence formula (3.23) we easily obtain that F_n is an interval exchange transformation. ▌

Lemma 3.9. Suppose that α satisfies (1.4) and is fixed. Then with probability 1 for any natural number n, F_n is minimal.

Proof. For $n=1$ the statement trivially follows from (3.26) and Assumption M(ii). For $n \geqslant 2$ we have the following representation:

$$F_n = S_{\xi_n} \circ \tilde{F}_{n-1} \circ S_{\xi_n} \tag{3.27}$$

where $\tilde{F}_{n-1} = I(n)^{-1} \circ F''_{n-1} \circ I(n)$ in accordance with (3.23) and Lemma 3.8 is an interval exchange transformation that depends only on h_j, r_j, $1 \leqslant j \leqslant n-1$. It is obvious that the minimality of F_n is equivalent to the minimality of the $S_{\xi_n} \circ F_n \circ S_{\xi_n}^{-1} = S_{\xi_n}^2 \circ \tilde{F}_{n-1}$. Since, in view of Assumption M, ξ_n and $\tilde{F}_{n-1}$ are independent, applying Theorem 3, we obtain with probability 1 the minimality of F_n. ▌

Lemma 3.10. For any natural number n and fixed α satisfying (1.4) with probability 1 the set $D_{n,x} = \{G_n^k x, k \in \mathbb{Z}_+\}$ is dense in $\mathscr{L}_n$ for any $x \in \mathscr{L}_n$.

Proof. If $n=1$, the statement trivially follows from Lemma 3.9. Assume the statement to be true for $n-1$. For the beginning we assume that the starting point x belongs to $\hat{L}^u(n)$. We use the notations introduced to derive the recurrence formula (3.23) in considering the surfaces $\hat{R}^{(n)}$ and $\hat{R}^{(n-1)}$ and the corresponding mappings. If $\bar{A}$ is the closure of A, then, referring to Lemma 3.9, we can state with probability 1 that

$$\bar{D}_{n,x} \cap \hat{L}^u(n) = \hat{L}^u(n), \qquad \forall x \in \hat{L}^u(n) \tag{3.28}$$

In other words, if $(\Omega, \mathscr{F}, P)$ is our probability space, then there exist such Ω_1 that $P(\Omega_1)=1$ and for any $\omega \in \Omega_1$, (3.28) is true. From (3.28), (3.17), and the obvious inclusion $G_n(D_{n,x}) \subseteq D_{n,x}$ we have

$$\bar{D}_{n,x} \cap \hat{L}^l(n) = \hat{L}^l(n), \qquad \forall x \in \hat{L}^u(n) \tag{3.29}$$

and, in particular,

$$\bar{D}_{n,x} \cap C^l_-(n) = C^l_-(n), \qquad \forall x \in \hat{L}^u(n) \tag{3.30}$$

The equalities (3.28) and (3.29) mean that the set $D_{n,x}$ is dense in the sets $\hat{L}^l(n)$ and $\hat{L}^u(n)$, and therefore it remains to prove that this set is dense in other lines of $\mathscr{L}_n$. Let z be an arbitrary point on $\bigcup_{j=0}^{n-1} (\hat{L}^l(n) \cup \hat{L}^u(n))$. In view of (3.4), we may treat z as a point in $\mathscr{L}'_{n-1}$. Denote by E_{n-1} the subset of $\mathscr{L}_{n-1}$ that consists of points having the form $G^k_{n-1}e$, where e is a endpoint of intervals from $C^l(j)$, $C^u(j)$, $1 \leqslant j \leqslant n-1$, and k is a non-negative integer. Since E_{n-1} is a countable set, we can choose y such that

$$y \in C^u(n-1), \qquad y \notin E_{n-1} \tag{3.31}$$

Let ε be any positive real number. Because of the assumption about $R^{(n-1)}$, we can find such a Ω_2 that $P(\Omega_2) = 1$ and for any ω from Ω_2, (3.28) is true, and any trajectory $G^k_{n-1}x$, $k \in \mathbb{Z}_+$, is dense in $\mathscr{L}_{n-1}$ for any $x \in \mathscr{L}_{n-1}$. In particular, for y satisfying (3.31) we can find such a natural number l that

$$|G^l_{n-1}y - z| < \varepsilon \tag{3.32}$$

Now, if $K = \{k \in \mathbb{Z}_+ : 1 \leqslant k \leqslant l,\ G^k_{n-1}y \in C^u(n-1)\}$ and $m = \max\{k : k \in K\}$, then we have

$$y_1 = G^m_{n-1}y \in C^u(n-1), \qquad G^k_{n-1}y_1 \notin C^u(n-1), \qquad 1 \leqslant k \leqslant l-m \tag{3.33}$$

and

$$|G^{l-m}_{n-1}y_1 - z| < \varepsilon \tag{3.34}$$

Notice that from (3.33) and (3.34) it follows that the trajectory of the point $y_1 \in C^u(n-1)$ gets close to z being within $R^{(n-1)}$. This means that using isomorphisms $J(n)$, $J_-(n)$, we may write

$$G^{l-m}_{n-1}y_1 = G^{l-m}_n \circ J^{-1}_-(n) \circ J^{-1}(n)\, y_1$$

and therefore

$$|G^{l-m}_n \circ J^{-1}_-(n) \circ J^{-1}(n)\, y_1 - z| < \varepsilon \tag{3.35}$$

Now, because of the choice of y as satisfying (3.31), and the second relation in (3.33), we can find a small interval $I(y_1) \subset C^u(n-1)$, containing y_1, such that

$$|G^{l-m}_n \circ J^{-1}_-(n) \circ J^{-1}(n)\, y' - z| < 2\varepsilon, \qquad y' \in I(y_1) \tag{3.36}$$

In particular, in view of (3.29), we can take $y' \in I(y_1)$ such that

$$y'' = J_-^{-1}(n) \circ J^{-1}(n)\, y' \in D_{n,x} \cap \hat{L}^l(n)$$

Since $G_n^{l-m} y'' \in D_{n,x}$, (3.36) means that the points of $D_{n,x}$ can by arbitrarily close to z, or, finally, that $D_{n,x}$ is dense in $\mathscr{L}_n$. Let us recall that we proved the statement assuming that the starting point x is on $\hat{L}^u(n)$. In the general case the previous reasoning [in particular, the validity of the statement for $R^{(n-1)}$ and (3.15)] shows that the series of points $G_n^k x$, $k \in \mathbb{Z}_+$, must contain some point $G_n^{k_0} x$ that belongs to $\hat{L}^u(n)$. In other words, this means that the ray must sooner or later leave the surface $R^{(n-1)}$. This reduces the general case of the starting point to the considered one, and completes the proof of the lemma. ▌

Proof of Theorem 1.2 and Theorem 1.3. Theorem 1.3 is a straightforward corollary of Lemma 3.10. Indeed, as follows from this lemma with probability 1, the trajectory of light is dense in $\mathscr{L}_n$. Using Proposition 3.1 and the construction of $\mathscr{L}_n$ and $R^{(n)}$, we can easily see that there exists a positive T such that

$$\bigcup_{0 \leqslant t \leqslant T} V^t \mathscr{L}_n = \hat{R}^{(n)}$$

This obviously leads to the density of the trajectory in the whole surface $R^{(n)}$. Then, having proved Theorem 1.3, we notice the following: if the ray has a fixed angle with respect to the y axis, say in the rectangle R_0, and any starting point, then in accordance with Theorem 1.3 with probability 1 the ray must sooner or later leave any system of a finite number of connected rectangles. This obviously leads to the statement of Theorem 1.2.

Remarks. 1. Some models for which a wind particle can be trapped and for which there is no diffusion are considered in ref. 5.

2. For the construction of the Riemann surfaces for rational polygons see refs. 6 and 7.

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